Kuwait University

Department of Mathematics — Final examination, 5 June 2011

Math 101: Calculus I — Duration 120 minutes

Answer all of the following questions.
Calculators and mobile telephones are not allowed.

1. (a) [2 pts.] Find $\lim_{x\to\infty}\left(2x-\sqrt{4x^2+5x}\right)$.

   (b) [2 pts.] Find $\lim_{x\to 2}\dfrac{x\sin(x-2)}{x^2-3x+2}$.

2. [4 pts.] Let

$$f(x)=\begin{cases} ax+b & \text{for } x\le 0\\ x^2+a-b & \text{for } 0<x\le 2\\ \cos(x-2) & \text{for } x>2.\end{cases}$$

   Find the values of a and b for which f is continuous on $(-\infty,\infty)$.

3. [4 pts.] Let k be any real number. Show that the equation $x^3-15x+k=0$ has at most one root in $[-2,2]$.

4. [4 pts.] Find the value of $m<0$ that minimizes the area of the region bounded by the x-axis, the y-axis, and the line $y=mx+5-3m$.

5. [4 pts.] The curve $y=f(x)$ has slope $f'(x)=\sqrt{4x+1}+x\sin(x^2)+1$, and passes through the point $(0,1)$. Find f.

6. [4 pts.] Prove that

$$\frac{3}{4}<\int_{1/2}^{2}\frac{1}{x}\,dx<3.$$

7. [4 pts.] Find the smallest positive critical number of

$$f(x)=\int_0^{x^2}\cos(t^{3/2})\,dt.$$

8. [4 pts.] Find the area of the region bounded by the graphs of $y=8/x^2$, $y=8x$, and $y=x$.

9. [4 pts.] Find the volume of the solid obtained by the rotating the region bounded by $y=\sqrt{x}$, the x-axis, and $x=4$ about the line $x=5$.

10. Let $f(x)=\sqrt{x}$.

    (a) [2 pts.] Find the average value of f on the interval $[4,9]$.

    (b) [2 pts.] Find the number c that satisfies the conclusion of the Mean Value Theorem for Integrals for the function f on $[4,9]$.

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1. (a) For $x > 0$,

$$\begin{aligned} 2x - \sqrt{4x^2+5x} &= \left(2x - \sqrt{4x^2+5x}\right) \frac{2x + \sqrt{4x^2+5x}}{2x + \sqrt{4x^2+5x}} \\ &= -\frac{5x}{2x + \sqrt{4x^2+5x}} = -\frac{5}{2 + \sqrt{4+5/x}}. \end{aligned}$$

Hence,

$$\lim_{x\to\infty} \left(2x - \sqrt{4x^2+5x}\right) = -\frac{5}{2+\sqrt{4+0}} = -\frac{5}{4}.$$

(b)

$$\frac{x \sin(x-2)}{x^2-3x+2} = \frac{x}{x-1} \cdot \frac{\sin(x-2)}{x-2}.$$

So,

$$\lim_{x\to 2} \frac{x \sin(x-2)}{x^2-3x+2} = \frac{2}{2-1} \cdot 1 = 2.$$

2. The functions $ax + b$, $x^2 + a - b$, and $\cos(x-2)$ are continuous everywhere. Therefore, one only needs to check that f is continuous from the right at 0 and 2, i.e.

$$f(0) = b = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 + a - b = a - b$$

and

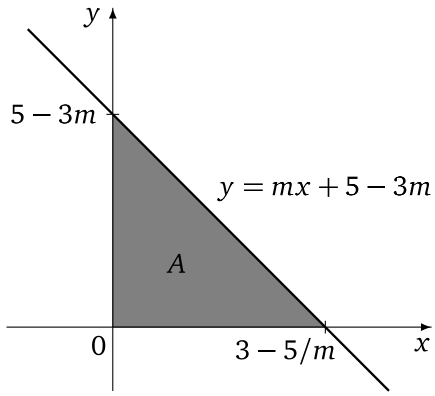
$$f(2) = 4 + a - b = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \cos(x-2) = 1.$$

The equations

$$\begin{cases} b = a - b \\ 4 + a - b = 1 \end{cases} \implies \cdots \implies \begin{cases} a = -6 \\ b = -3. \end{cases}$$

3. Let $f(x) = x^3 - 15x + k$. Suppose that the equation $f(x) = 0$ has two roots $a < b$ in the interval $[-2,2]$. Then f is continuous on $[a,b]$, differentiable on (a,b), and $f(a) = f(b) = 0$. Hence by Rolle's Theorem, there is a $c \in (a,b)$ such that $f'(c) = 0$. However, $f'(x) = 3x^2 - 15 < 3(2^2) - 15 = -3$ for all $x \in (-2,2)$. Thus, by contradiction, there cannot be two roots in the interval $[-2,2]$.

4. The line $y = mx + 5 - 3m$ intercepts the x-axis when $x = 3 - 5/m$.
The line $y = mx + 5 - 3m$ intercepts the y-axis when $y = 5 - 3m$.



Thus the area A of the region, which is a triangle, is

$$A = \frac{1}{2}\left(3 - \frac{5}{m}\right)(5-3m) = \frac{1}{2}\left(30 - 9m - \frac{25}{m}\right).$$

This implies

$$\frac{dA}{dm} = \frac{1}{2}\left(-9 + \frac{25}{m^2}\right) = \cdots = -\frac{9}{2m^2}\left(m - \frac{5}{3}\right)\left(m + \frac{5}{3}\right).$$

Hence A has a critical number $m < 0$ if and only if $m = -5/3$.

$$\frac{d^2A}{dm^2} = -\frac{25}{m^3} > 0 \quad \text{for } m < 0.$$

So by the Second Derivative Test, $m = -5/3$ is a minimum.
Answer: $m = -5/3$.

5. By the substitution $u = 4x+1$, there holds $\int \sqrt{4x+1}\,dx = \int u^{1/2}\,\frac{1}{4}du = \frac{1}{6}u^{3/2} + C$.
By the substitution $v = x^2$, there holds $\int x \sin x^2\,dx = \int \sin v\,\frac{1}{2}dv = -\frac{1}{2}\cos v + C$.
Hence,

$$f(x) = \tfrac{1}{6}(4x+1)^{3/2} - \tfrac{1}{2}\cos(x^2) + x + C$$

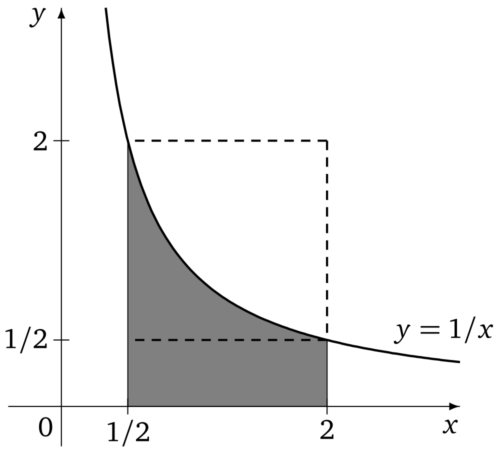
for some number C. Because $f(0) = 1$,

$$1 = \tfrac{1}{6}[4(0)+1]^{3/2} - \tfrac{1}{2}\cos(0^2) + 0 + C \quad \Longrightarrow \quad \cdots \quad \Longrightarrow \quad C = \frac{4}{3}.$$

Answer:

$$f(x) = \tfrac{1}{6}\left[(4x+1)^{3/2} - 3\cos(x^2) + 6x + 8\right] \quad \text{for } x \geqslant -1/4.$$

6. If $1/2 < x < 2$ then $1/2 < 1/x < 2$.



So,

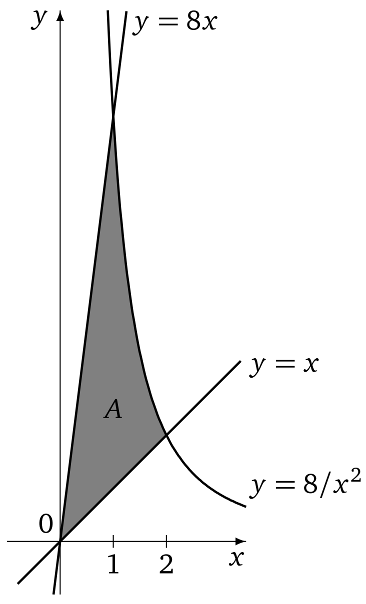
$$\frac{3}{4} = \left(2 - \frac{1}{2}\right)\frac{1}{2} < \int_{1/2}^{2} \frac{1}{x}\,dx < \left(2 - \frac{1}{2}\right)2 = 3.$$

7. Let $u = x^2$. Then, for $x > 0$,

$$f'(x) = \frac{d}{dx}\int_0^u \cos(t^{3/2})\,dt = \left(\frac{d}{du}\int_0^u \cos(t^{3/2})\,dt\right)\frac{du}{dx} = \cos(u^{3/2})2x = 2x\cos x^3.$$

   Hence, the smallest positive critical number is given by the smallest positive root of $\cos x^3 = 0$, i.e. $x = \sqrt[3]{\pi/2}$.

8. The curve $y = 8/x^2$ and the line $y = 8x$ intersect when $8/x^2 = 8x$, i.e. $x = 1$ and $y = 8$.
   The curve $y = 8/x^2$ and the line $y = x$ intersect when $8/x^2 = x$, i.e. $x = 2$ and $y = 2$.
   The lines $y = 8x$ and $y = x$ intersect when $8x = x$, i.e. $x = 0$ and $y = 0$.



The area

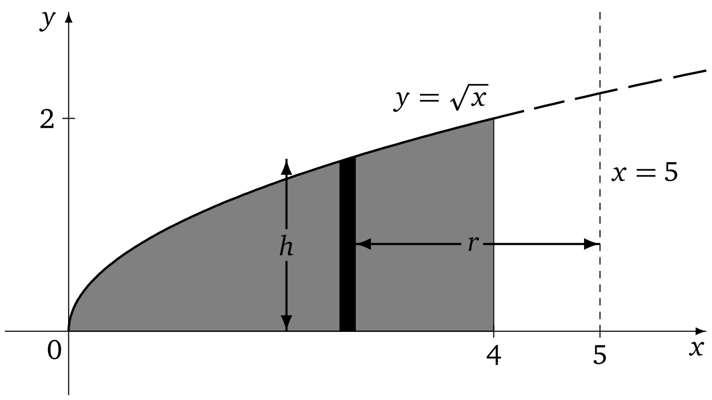
$$\begin{aligned} A &= \int_0^1 (8x - x)\,dx + \int_1^2 \left(\frac{8}{x^2} - x\right)\,dx = \Big[7x\Big]_0^1 dx + \left[-\frac{8}{x} - \frac{x^2}{2}\right]_1^2 \\ &= 7(1) - 7(0) + \left(-\frac{8}{2} - \frac{2^2}{2}\right) - \left(-\frac{8}{1} - \frac{1^2}{2}\right) = \cdots = 6. \end{aligned}$$

9. By the method of cylindrical shells, the volume

$$V = \int_0^4 2\pi r h\,dx$$

where

$$r = 5 - x \quad \text{and} \quad h = \sqrt{x}.$$



So,

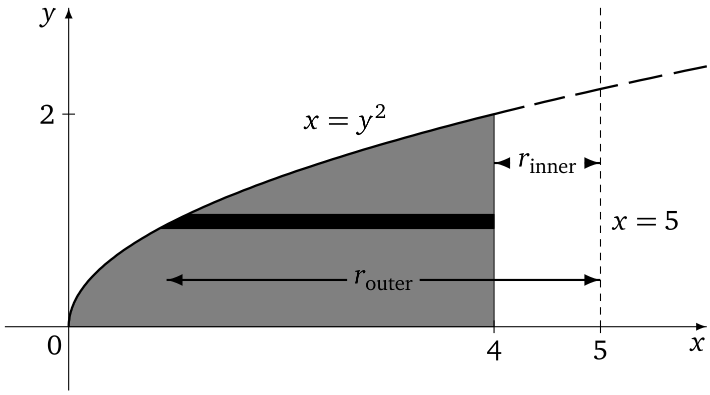
$$\begin{aligned} V &= \int_0^4 2\pi(5-x)\sqrt{x}\,dx = 2\pi \int_0^4 \left(5x^{1/2} - x^{3/2}\right)\,dx = \pi\left[\frac{10}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^4 \\ &= 22\pi\left(\tfrac{10}{3}4^{3/2} - \tfrac{2}{5}4^{5/2}\right) = \cdots = \tfrac{416}{15}\pi. \end{aligned}$$

By the method of disks and washers,

$$V = \int_0^2 \left[\pi(r_{\text{outer}})^2 - \pi(r_{\text{inner}})^2\right]\,dy$$

where

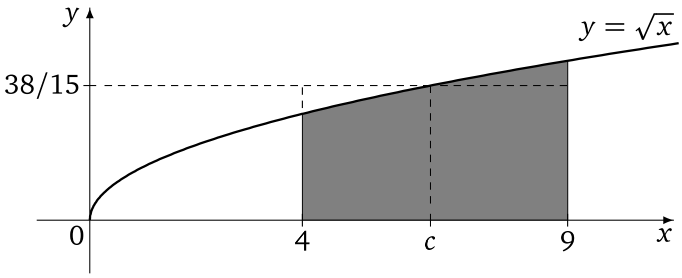
$$r_{\text{outer}} = 5 - y^2 \quad \text{and} \quad r_{\text{inner}} = 1.$$



So,

$$\begin{aligned} V &= \int_0^2 \left[\pi(5-y^2)^2 - \pi 1^2\right] dy = \pi \int_0^2 (y^4 - 10y^2 + 24)\, dy \\ &= \pi \left[\tfrac{1}{5}y^5 - \tfrac{10}{3}y^3 + 24y\right]_0^2 = \pi \left[\tfrac{1}{5}2^5 - \tfrac{10}{3}2^3 + 24(2)\right] = \cdots = \tfrac{416}{15}\pi. \end{aligned}$$

10.



(a) The average value is

$$\frac{1}{9-4}\int_4^9 f(x)\, dx = \frac{1}{5}\left[\frac{2}{3}x^{3/2}\right]_4^9 = \frac{2}{15}\left(9^{3/2} - 4^{3/2}\right) = \cdots = \frac{38}{15}.$$

(b) The number c is such that

$$\sqrt{c} = f(c) = \frac{1}{9-4}\int_4^9 f(x)\, dx = \frac{38}{15}.$$

So $c = (38/15)^2 = 1444/225$.